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Approximation of the ruin probability using the scaled Laplace transform inversion

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Abstract

The problem of recovering the ruin probability in the classical risk model based on the scaled Laplace transform inversion is studied. It is shown how to overcome the problem of evaluating the ruin probability at large values of an initial surplus process. Comparisons of proposed approximations with the ones based on the Laplace transform inversions using a fixed Talbot algorithm as well as on the ones using the Trefethen–Weideman–Schmelzer and maximum entropy methods are presented via a simulation study.

Keywords

Classical risk model; Ruin probability; Moment-recovered approximation; Laplace transform inversion; Uniform rate of approximation

1. Introduction

Recovering a function from its Laplace transform represents a very severe ill-posed inverse problem (Tikhonov and Arsenin [1]). That is why the regularization is very helpful in situations using the Laplace transform inversion. In Mnatsakanov et al. [2], the regularized inversion of the Laplace transform (Chauveau et al. [3]) has been used for approximation as well as for estimation of the ruin probability. Under the conditions on ruin probability, the upper bound for integrated squared error was derived, and rate of convergence of order 1/log n was obtained in the classical risk model. See also Shimizu [4] for application of this approach in estimating the expected discount penalty function in the Lévy risk model. Our simulation study shows that the approximation rate derived in Mnatsakanov et al. [2] is not optimal. This motivated our interest to improve the rate using the scaled Laplace transform inversion suggested by Mnatsakanov and Sarkisian [5].

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There are many approaches which deal with approximating the ruin probability ψ. See, for example, Gzyl et al. [6], Avram et al. [7], and Zhang et al. [8] among others. A very interesting approximation based on the Trefethen–Weideman–Schmelzer (TWS) method (see [9]) is constructed in Albrecher et al. [10]. In the latter work the authors assume that the claim size distribution represents a completely monotone function.

Note that there are several difficulties associated with inverting the Laplace transform. For example, when applying the Padé approximation one cannot guarantee the positivity of the obtained approximation of ψ, and its rate of convergence. Gzyl et al. [6] applied the maximum entropy (ME) method to approximate ψ. In order to reduce the ill-conditioning of ME approximation, the authors used the fractional moments of the exponential transform and derived an accurate approximation of ψ by reducing the ruin problem to the Hausdorff moment problem on [0, 1].

It is worth mentioning that the moment-recovered (MR) constructions proposed in Section 2 also enable us to reduce the Stieltjes moment problem to the Hausdorff one (cf. with Corollary 1 (iii), as well as Corollary 4 (ii)–(iii) in Mnatsakanov [11] and [12], respectively).

Note also that the Laplace transform inversions proposed in [5–12] do not require the claim size distribution F to be a completely monotone function. See, for example, Gzyl et al. [6], where the ME method and MR-approach proposed in Mnatsakanov [12] are compared. In particular, the cases with gamma (a, β) (with $a > 1$) and uniform on [0, 1] claim size distributions are considered. To conduct the comparison the authors used the formula from [12] (see also (A.4) in the Appendix) with the number of integer moments $\alpha = 60$. In the case of the gamma (2, 1) model we show that the construction (17) has a better performance in terms of *sup*-norm when compared to the ME counterpart from [6] when $\alpha \geq 60$ and the optimal scaling parameter $1 < b \leq e$. As a result we obtained an accurate approximation of the ruin probability ψ (see Fig. 2(a) in Section 4). Calculations for large values of the parameter $\alpha \geq 60$ have been performed using a new programming language called SmartXML being developed by Artak Hakobyan (see his web page: www.oroptimizer.com). The calculations performed using SmartXML avoid many numerical problems, and perform very well for models related to the Hausdorff, Stieltjes, and Hamburger moment problems.

The main aim of present article is to derive the upper bound for the rate of MR-approximation (4) of a function f in *sup*-norm and demonstrate its performance in the ruin problem via a simulation study. We show that the MR-construction of ruin probability, see (17) below, with the large value of α and appropriately chosen parameter b, performs better than its ME counterpart (cf. with [6]) and is comparable with the ones using TWS and a fixed Talbot (FT) algorithms (cf. with [10]).

The remainder of this article is organized as follows. In Section 2 we introduce two MR-approximations of a function and its derivative, see (3) and (4), respectively, and provide the upper bound for MR-approximation (4); the upper bound for (3) has been already established in [5] (see also Theorem 1A in the Appendix). In Section 3, we propose two approximations (see (17) and (20)) of the ruin probability ψ, which are based on the finite number of values of Laplace ($\mathscr{L}_\psi$) and Laplace–Stieltjes (L_G) transforms of ψ and $G = 1 - \psi$, respectively:

$$\mathscr{L}_\psi(s)=\int_0^\infty e^{-s\,\tau}\,\psi(\tau)\,d\tau \quad (1)$$

$$L_G(s)=\int_0^\infty e^{-s\,\tau}\,dG(\tau),\;\; s\in[0,\infty):=\mathbb{R}_+. \quad (2)$$

In Section 4 we consider two models (gamma and log-normal) in order to make a comparison with the approaches developed in [6] and [10]. In Examples 1 and 2 the performance of the proposed approximations are demonstrated graphically via Figs. 1 and 2 and Tables 1–4. In the case of the gamma (2, 1) model, for several values of α and the scaling parameter b, the maximum deviations between approximants and the true ruin probability are recorded in Table 1. When the claim size distribution is gamma, we compared the MR-approximations (17) and (20) with the approximant ψ_{FT} that is based on FT algorithm.

In the case when the claim size distribution is log-normal (Example 3), several values of MR-approximation $\psi_{\alpha,b}$ defined in (17) are compared with those of ψ_{TWS} (see Table 5 below and Table 9 in [10]).

Finally, in the Appendix, we recall two results derived in [5] and [12], where the rates of MR-approximations of a cumulative distribution function (cdf) $F : \mathbb{R}_+ \to [0, 1]$ as well as the Laplace transform inversions in bivariate and univariate cases are presented.

2. Moment-recovered Laplace transform inversion

In this section we consider two approximations of the Laplace transform inversions recovering cdf F and its derivative f. Namely, let us suppose that a random variable X has a cdf F which is absolutely continuous with respect to the Lebesgue measure μ and has a support in $\mathbb{R}_+$.

To recover $\bar{F} = 1 - F$ and f consider:

$$\overline{F}_{\alpha,b}(x):=\left(L^{-1}_{\alpha,F}L_F\right)(x)=\sum_{k=0}^{[\alpha b^{-x}]}\sum_{j=k}^{\alpha}\binom{\alpha}{j}\binom{j}{k}(-1)^{j-k}L_F(j\ln b),\qquad x\in\mathbb{R}_+ \quad (3)$$

and

$$f_{\alpha,b}(x):=(\mathscr{L}^{-1}_{\alpha,f}\mathscr{L}_f)\,(x)=\frac{b^{-x}\ln b\,\Gamma(\alpha+2)}{\Gamma([\alpha b^{-x}]+1)}\sum_{j=0}^{\alpha-[\alpha b^{-x}]}\frac{(-1)^j\mathscr{L}_f((j+[\alpha b^{-x}])\ln b)}{j!(\alpha-[\alpha b^{-x}]-j)!}, \quad (4)$$

respectively. Here we assume that $\alpha \in \mathbb{N}$ is an integer-valued parameter, and the scaling parameter $b \in (1, e]$. In other words, to evaluate the approximations $\bar{F}_{\alpha,b}$ and $f_{\alpha,b}$, only the knowledge of the Laplace and Laplace–Stieltjes transforms evaluated at the finite arithmetic progression $\mathbb{N}_{\alpha,b} = \{j \ln b, j = 0, 1, \ldots, \alpha\}$ are required. Also, note that construction (3) has been proposed in [5] (Eq. (18)) and applied for approximation of a compound Poisson distribution as well as in a decompounding problem.

There are several other approximations of the Laplace transform inversion that use the Laplace transform evaluated at the set of finite or infinite values of argument of $\mathscr{L}_f$. We refer to [6] where the nested minimization procedure was used in combination with the ME method to solve the ruin problem. Their construction is based on choosing the optimal α_j-fractional moments ($j = 1, \ldots, M$) of the random variable $Y = e^{-X}$ given the countable sequence of ordinary moments of Y. Application of the ME procedure to the distribution of $Y = e^{-X}$ is very helpful since it reduces the ruin problem to the Hausdorff moment problem on [0, 1], and provides a more stable procedure. We also refer to [13] where the ME method is applied to fractional moments in the Hausdorff case. The idea of using the transformations including the logarithmic one has been applied in other papers as well [5,12,14,15]. In [15] the sequence of fractional moments of Y of orders $\{\alpha_j, j \in \mathbb{N}\}$ is used to invert the Laplace transform. Their technique is based on the orthogonal projections of the underlying function on the space generated by an orthogonal system of Muntz polynomials. Besides, they require the sequence $\{\alpha_j, j \in \mathbb{N}\}$ to obey the following condition:

$$\sum_{j=0}^{\infty}\frac{2\alpha_j-1}{(2\alpha_j-1)^2+1}=\infty, \quad \text{where} \quad \alpha_j>1/2.$$

Actually, in the current work we also apply the above mentioned transformation and extend the MR-constructions ([12], see also (A.4) and (A.5) in the Appendix) to the case when the set of moments of non-integer orders $\mathbb{N}_{\alpha,b}$ of Y is used instead of the set $\mathbb{N}_{\alpha} = \{0, 1, \ldots, \alpha\}$ of integer orders. On the other hand, compared to the ME method in [6] and the orthogonal projection technique in [15], the constructions (3) and (4) provide direct analytical forms based on the first α values of the scaled Laplace transforms and depend only on appropriately chosen parameter $b \in (1, e]$.

In the following, by "$\to_w$" we mean the weak convergence of cdfs (i.e., convergence at each continuity point of the limiting cdf), while the uniform convergence will be denoted by "$\to_u$", and by $\|f\|$ the *sup*-norm of f.

The upper bound in *sup*-norm for MR-approximation $\bar{F}_{\alpha,b}$ has been derived in [5] (see also Theorem 1A in the Appendix). To establish the rate of approximation for $f_{\alpha,b}$ let us recall the result from Mnatsakanov [11]:

Let Y be random variable with cdf Q defined [0, 1], and $q = Q'$ be its density. Assume that the moment sequence of Q (up to order α) is available:

$$m(j)=\int_0^1 t^j\,dQ(t), \quad j \in \mathbb{N}_\alpha. \quad (5)$$

Define the MR-approximation of q by means of $q_\alpha:=\mathscr{B}_\alpha^{-1}\,\nu$, where

$$\left(\mathscr{B}_{\alpha}^{-1}\nu\right)(x)=\frac{\Gamma(\alpha+2)}{\Gamma([\alpha x]+1)}\sum_{j=0}^{\alpha-[\alpha x]}\frac{(-1)^{j}m(j+[\alpha x])}{j!(\alpha-[\alpha x]-j)!},\quad x\in[0,1]. \quad (6)$$

Theorem 1 ([11])

i. *If pdf q is continuous on* [0, 1], *then* $q_\alpha \to_u q$, *as* $\alpha \to \infty$.

ii. *If* q'' *is bounded on* [0,1], *then*

$$\|q_\alpha-q\| \le \frac{1}{\alpha+2}\left\{2\|q'\|+\frac{1}{2}\|q''\|+\frac{2}{\alpha+1}\|q''\|\right\}. \quad (7)$$

Remark 1

To recover a cdf Q the following approximation

$$B_{\alpha,Q}(x)=\sum_{k=0}^{[\alpha x]}\sum_{j=k}^{\alpha}\binom{\alpha}{j}\binom{j}{k}(-1)^{j-k}m(j),\quad x\in[0,1],$$

has been proposed and some of its properties were investigated in Mnatsakanov and Ruymgaart [16]. In particular, the relationship between $B_{\alpha,Q}$ and binomial mixture, as well as convergence $B_{\alpha,Q} \to_w Q$ were established: from Eqs. (3.3) and (3.9) we conclude

$$B_{\alpha,Q}(x)=\int_0^1 \mathbb{P}(\mathrm{B}(\alpha,t)\le[\alpha x])\,dQ(t)\to Q(x)\quad \text{as}\quad \alpha\to\infty,$$

for any continuity point x of Q. Here by $B(\alpha, t)$ we denote a binomial random variable with parameters α and t. In other words, for the binomial random variable $B(\alpha, Y)$ with random parameter Y we have the following relationship:

$$\mathbb{P}(B(\alpha,y)\le[\alpha x])\to Q(x)\quad \text{as } \alpha\to\infty.$$

Finally, note that if the sample $Y_1, \ldots, Y_n$ from Q is available, then the empirical counterpart of $B_{\alpha,Q}(x)$ provides the estimate of $Q(x)$:

$$\widehat{B}_{\alpha,Q}(x)=\frac{1}{n}\sum_{i=1}^{n}\sum_{k=0}^{[\alpha x]}\binom{\alpha}{k}Y_i^k(1-Y_i)^{\alpha-k}$$

(cf. with Eq. (13) in [5]).

Now, assume that $Q(u) = F(\varphi^{-1}(u))$, where $\varphi^{-1}(u)$ is an inverse function of some decreasing function $\varphi : \mathbb{R}_+ \to [0, 1]$. Note that

$$q(u)=Q^{'}(u)=\frac{f(\phi^{-1}(u))}{\phi^{'}(\phi^{-1}(u))}:=\frac{f}{\phi^{'}}\circ\phi^{-1}(u), q^{'}(u)=\frac{f^{'}}{(\phi^{'})^{2}}\circ\phi^{-1}(u)-\frac{f\phi^{''}}{(\phi^{'})^{3}}\circ\phi^{-1}(u),$$
$$q^{''}=\frac{f^{''}}{(\phi^{'})^{2}}\circ\phi^{-1}-\frac{f^{'}\phi^{''}}{(\phi^{'})^{3}}\circ\phi^{-1}-\frac{f^{'}\phi^{''}}{(\phi^{'})^{4}}\circ\phi^{-1}-\frac{f\phi^{''^{'}}}{(\phi^{'})^{4}}\circ\phi^{-1}+3\frac{f\phi^{''}}{(\phi^{'})^{5}}\circ\phi^{-1}. \quad (8)$$

Consider the following conditions:

$$M_2=\|\frac{f\phi^{''}}{(\phi^{'})^3}\|<\infty, \quad \text{and} \quad M_3=\|\frac{f^{'}}{(\phi^{'})^2}\|<\infty, \quad (9)$$

$$M=\sum_{j=4}^{8}M_j<\infty, \quad \text{where} \quad M_4=\|\frac{f^{''}}{(\phi^{'})^2}\|, \quad M_5=\|\frac{f^{'}\phi^{''}}{(\phi^{'})^3}\|,$$
$$M_6=\|\frac{f^{'}\phi^{''}}{(\phi^{'})^4}\|, \quad M_7=\|\frac{f\phi^{''^{'}}}{(\phi^{'})^4}\|, \quad M_8=3\|\frac{f\phi^{''}}{(\phi^{'})^5}\|. \quad (10)$$

Let $\nu_\varphi = \{m_\varphi(j) = \int[\varphi(t)]^j f(t)dt, j \in \mathbb{N}_\alpha\}$, $q \doteq -q$ with $q(u)=\frac{f}{\phi^{'}}\circ\phi^{-1}(u)$, and $\overline{q}_\alpha=\mathscr{B}_\alpha^{-1}\nu_\phi$ with

$$\left(\mathscr{B}_\alpha^{-1}\nu_\phi\right)(x)=\frac{\Gamma(\alpha+2)}{\Gamma([\alpha x]+1)}\sum_{j=0}^{\alpha-[\alpha x]}\frac{(-1)^j m_\phi(j+[\alpha x])}{j!(\alpha-[\alpha x]-j)!}, \quad x\in[0,1]. \quad (11)$$

In the next statement we establish a new upper bound that is valid for any continuous and decreasing function $\varphi : \mathbb{R}_+ \to [0, 1]$ satisfying conditions (9) and (10). Namely,

Lemma 1

- **i.** *Assume f and* φ *are both continuous functions on* $\mathbb{R}_+$, *then*

$$\overline{q}_a=\mathscr{B}_\alpha^{-1}\nu_\phi\rightarrow_u\overline{q} \ \ on \ [0,1], \ as \ \alpha\to\infty.$$

- **ii.** *Assume f has up to second order finite derivatives,* φ *has derivatives up to order three, and the conditions* (9)–(10) *are satisfied. Then*

$$\|\overline{q}_\alpha-\overline{q}\|\leq\frac{1}{\alpha+1}\left\{2M_2+2M_3+\frac{1}{2}M\right\}+o\left(\frac{1}{\alpha}\right) \quad as \quad \alpha\to\infty.$$

Proof of Lemma 1—Note that

$$m_\phi(j)=\int_0^\infty[\phi(t)]^j f(t)dt=-\int_0^1 u^j q(u)du=-m(j). \quad (12)$$

Hence, combining (11) and (12) with (5) and (6) and the statement Theorem 1 (*i*), we obtain (*i*).

The statement (*ii*) follows from (8) to (10) and Theorem 1 (*ii*), since according to (7) we have

$$\|\overline{q}_\alpha - \overline{q}\| = \|q_\alpha - q\| \le \frac{1}{\alpha+2}\left\{2\|q^{'}\| + \frac{1}{2}\|q^{''}\| + \frac{2}{\alpha+1}\|q^{''}\|\right\}.$$

Consider the following conditions:

$$A_1 = \sup_{x>0}|f(x)b^{4x}| < \infty, \quad A_2 = \sup_{x>0}|f^{'}(x)b^{3x}| < \infty, \quad A_3 = \sup_{x>0}|f^{''}(x)b^{2x}| < \infty. \quad (13)$$

In the following statement, we use the generic constant C_k, $k = 0, 1, 2$, that can be easily identified by means of the constants A_k, $k = 1, 2, 3$, and conditions (9)–(10), where $\varphi(x) = b^{-x}$, $x \in \mathbb{R}_+$, with some $b > 1$.

Theorem 2

Assume f is a continuous function on $\mathbb{R}_+$ *and* $\varphi(x) = b^{-x}$ *for some* $b > 1$.

i. *Then* $f_{\alpha,b} \to_u f$;

ii. *If f has up to second order finite derivatives and conditions (*13*) are satisfied, then*

$$\|f_{\alpha,b} - f\| \le \frac{1}{\alpha+1}\left\{C_0 + \frac{C_1}{\ln b} + \frac{C_2}{\ln^2 b}\right\} + o\left(\frac{1}{\alpha}\right) \quad as \quad \alpha \to \infty. \quad (14)$$

Proof of Theorem 2—follows directly from Lemma 1. We only need to note that for $\varphi(x) = e^{-x\ln b}$, $x \in \mathbb{R}_+$, with some $b > 1$, we have $\varphi'(x) = -b^{-x}\ln b$, which, in combination with Lemma 1 (*i*) and (4), provides (*i*):

$$f_{\alpha,b}(x) = |\phi^{'}(x)|\overline{q}_\alpha(\phi(x)) \to |\phi^{'}(x)|\overline{q}(\phi(x)) = \phi^{'}(x)q(\phi(x)) = f(x)$$

uniformly on $\mathbb{R}_+$, as $\alpha \to \infty$. To prove (*ii*), let us mention that the functions f, f', and f'', satisfying conditions (13) also satisfy conditions (9) and (10) with $\varphi(x) = b^{-x}$, $b > 1$. Hence, application of Lemma 1 (*ii*) and (4) yields (14).

3. Approximation of the ruin probability

Let us mention another application of the moment-recovered construction (4) in the framework of a classical risk model. In actuarial literature, it is well known that the evaluation of the ruin probability $\psi = 1 - G$ with

$$G(u) = \mathbb{P}\left\{u + ct - \sum_{k=1}^{N(t)} X_k \ge 0, \ \text{for all}\ t \ge 0\right\}, \quad u \ge 0, \quad (15)$$

is a difficult problem when the distribution F of claim sizes X_k's does not follow the exponential model. Usually, the insurance company receives income from the policies at a constant rate $c > 0$ per unit time, and it is assumed (the classical risk model) that $N(t)$, $t \geq 0$, is a Poisson process with intensity $\lambda > 0$. Also, it is assumed that N is independent from the sequence $X_1, X_2, \ldots$. Here $u > 0$ is the initial capital of the company at time $t = 0$.

Now, let us consider the scaled, by $c = \ln b$, Laplace transform $\mathscr{L}_\psi(s \ln b)$ of ψ. One can apply the Pollaczek–Khinchine formula to obtain the following expression:

$$\mathscr{L}_\psi(s\ln b)=\frac{1}{s\ln b}-\frac{1-\rho}{s\ln b-\lambda_c\{1-\mathscr{L}_f(s\ln b)\}} \quad (16)$$

(cf. with Mnatsakanov et al. [2]). Here we assume $\rho = \lambda_c E(X) < 1$, and $\lambda_c := \lambda/c$.

To have a better graphical illustration of the MR-approximation (4), we suggest use of the factor $[\alpha b^{-x}]/\alpha$ instead of b^{-x} in the right-hand side of (4). Namely, consider

$$\psi_{\alpha,b}(x):=(\mathscr{L}_{\alpha,\psi}^{-1}\mathscr{L}_\psi)(x)=\frac{[\alpha b^{-x}]\ln b\,\Gamma(\alpha+2)}{\alpha\,\Gamma([\alpha b^{-x}]+1)}\sum_{j=0}^{\alpha-[\alpha b^{-x}]}\frac{(-1)^j\mathscr{L}_\psi((j+[\alpha b^{-x}])\ln b)}{j!(\alpha-[\alpha b^{-x}]-j)!} \quad (17)$$

(cf. with (A.4) from the Appendix).

It is worth mentioning that, given the sample $X_1, \ldots, X_n$ from F, we can estimate $\mathscr{L}_\psi$ in (16) by using the sample mean $\bar{X}$, $\hat{\rho} = \lambda_c \bar{X}$, and the empirical Laplace transform

$$\hat{\mathscr{L}}_f(s)=\frac{1}{n}\sum_{i=1}^{n}e^{-sX_i}.$$

Hence, applying $\hat{\mathscr{L}_\psi}$ instead of $\mathscr{L}_\psi$ in the right hand side of (17), we derive the estimate of ψ:

$$\hat{\psi}_{\alpha,b}(x):=(\mathscr{L}_{\alpha,\psi}^{-1}\hat{\mathscr{L}}_\psi)(x)=\frac{[\alpha b^{-x}]\ln b\,\Gamma(\alpha+2)}{\alpha\,\Gamma([\alpha b^{-x}]+1)}\sum_{j-0}^{\alpha-[\alpha b^{-x}]}\frac{(-1)^j\hat{\mathscr{L}}_\psi((j+[\alpha b^{-x}])\ln b)}{j!(\alpha-[\alpha b^{-x}]-j)!} \quad (18)$$

Here

$$\hat{\mathscr{L}}_\psi(s\ln b)=\frac{1}{s\ln b}-\frac{1-\hat{\rho}}{s\ln b-\lambda_c\{1-\hat{\mathscr{L}}_f(s\ln b)\}}. \quad (19)$$

Remark 2

Note that (3) recovers the survival function $\bar{F}$ given the Laplace–Stieltjes transform L_F of F. Hence, to approximate or estimate the ruin probability ψ one can also apply (3) where the

Laplace–Stieltjes transform L_G of $G = 1 - \psi$ is provided. In particular, combining (3) with L_G evaluated according to the formula (cf. with Dickson [17]):

$$L_G(j\ln b) = (j\ln b)\mathscr{L}_G(j\ln b) = \frac{(1-\rho)\, j\ln b}{j\ln b - \lambda_c\,\{1 - \mathscr{L}_f(j\ln b)\}},$$

we derive another approximate $\psi^*_{\alpha,b}$ of ψ. Namely,

$$\psi^*_{\alpha,b}(x) = \sum_{k=0}^{[\alpha b^{-x}]} \sum_{j=k}^{\alpha} \binom{\alpha}{j}\binom{j}{k}(-1)^{j-k} L_G(j\ln b), \qquad x \in \mathbb{R}_+ \quad (20)$$

(cf. with (A.5) from the Appendix). It is worth noting that from a simulation study we found out that in (20) (Table 4) as well as in (17) (Tables 2, 3 and 5) it is better to use the rounding value $[\alpha b^{-x}]$ instead of the integer part of αb^{-x}.

4. Examples

In this section we consider three examples. In Examples 1 and 2 we assume that the claim size distribution is specified as gamma (a, β) with two different pairs of the shape=a and scale=β parameters: $(a, \beta) \in \{(2, 1), (2.5, 0.4)\}$ and $\lambda_c \in \{0.2, 1.1\}$. In Example 3 we consider the log-normal (−1.62, 1.8) model with $\lambda = 1$ and $c \in \{1.05, 1.1, 1.15, 1.20, 1.25, 1.30, 2.0\}$. To present the smoothed (linearized) versions of the recovered ruin probabilities, we conducted our calculations by evaluating the values of $\psi_{\alpha,b}(x)$ and $\widehat{\psi_{\alpha,b}}(x)$ at $x \in \{(\ln\alpha - \ln(\alpha - j + 1))/\ln b,\ 1 \le j \le \alpha\}$ and plotted their linear interpolations in Fig. 1.

Example 1

Assume X has a gamma density $f(x) = \frac{x^{a-1}}{\beta^a \Gamma(a)} e^{-\frac{x}{\beta}}$ for $x > 0$. In this case the Laplace transform of f is

$$\mathscr{L}_f(s) = \left(\frac{1}{\beta\, s + 1}\right)^a$$

and $\mathscr{L}_\psi$ is defined as in (16). Consider the special case with $a = 2$, $\beta = 1$, $\lambda = 1$, and $c = 5$, i.e., $\lambda_c = 0.2$. This example is taken from Gzyl et al. [6], where the authors derived the exact form of true ruin probability

$$\psi(x) = 0.461862\, e^{-0.441742x} - 0.061862\, e^{-1.358257x} \quad \text{for} \quad x > 0. \quad (21)$$

Derivation of (21) is based on direct inversion of

$$\mathscr{L}_{\psi}(s)=\frac{2s+3}{5(r_1-r_2)}\left(\frac{1}{s-r_1}-\frac{1}{s-r_2}\right),$$

where $r_{1,2}=(-9+\sqrt{21})/10$ are the roots of quadratic equation $5s^2 + 9s + 3 = 0$. In [6], the MR-approximation (4) with $b = e$, i.e., (A.4) was evaluated as well. It should be mentioned that the value $b = e$ is recommended for use when the underlying function has a bounded support. The proposed approximation $\psi_{a,b}$ in (17) with $b < e$ behaves much better if compared to the one with $b = e$.

At first, we evaluated several values of $100 \times \|\psi_{a,b} - \psi\|$, when the range of a is specified as follows: $26 \le a \le 32$, and $b = 1.1 + 0.01k$, $0 \le k \le 20$. The pair $a = 27$, $b = 1.28$ was found to be the optimal one with corresponding value of $100 \times \|\psi_{a,b} - \psi\| = 0.60839$. In Fig. 1 we compared $\psi_{a,b}(x)$ (dotted curve) and $\hat{\psi}_{a,b}(x)$ (dotted curve) with the true ruin probability $\psi(x)$ (solid curve) when $b = 1.28$ and $b = 1.23$, respectively. In both plots we assumed $a = 27$.

To demonstrate how the accuracy of the approximation behaves when a increases, in Table 1 we recorded several values of $10^4 \times \|\psi_{a,b} - \psi\|$, when $a = 10\,k$, $k = 6, 9, 12, 15, 20, 40$, and $1.35 \le b \le 1.50$. Within this range of a, we found out that the pair $a = 400$, $b = 1.415$ can be considered as the optimal one since corresponding normalized absolute error $10^4 \times \|\psi_{a,b} - \psi\| = 1.3240$ is the smallest one. Let us compare $\psi_{a,b}$ with construction ψ_{EME} which is based on the elaborate ME (EME) method combined with the nested minimization procedure when the number of fractional moments $M = 8$. The values of ψ_{EME} have been cordially provided by Aldo Tagliani. In Fig. 2(a) we plotted the absolute errors of both approximations $\psi_{a,b}$ (solid and slashed curves) and ψ_{EME} (dotted curve) as the functions of initial surplus. From Fig. 2(a) we conclude that for chosen values of parameters, $a = 90$, 400, $b = 1.415$, the MR-approximate $\psi_{a,b}$ has a better performance in terms of the *sup*-norm if compared to ψ_{EME} when $M = 8$. In addition, in Table 2, several values of $\psi_{a,b}$ along with corresponding values of the approximant ψ_{FT} are recorded when $a = 5000$, $b = 1.4125$.We can say that by increasing a we improve the accuracy of $\psi_{a,b}$ considerably.

Example 2

Assume now that X_k's have gamma (2.5, 0.4) distribution. This example is taken from Albrecher et al. [10]. Here we compare the performances of $\psi_{a,b}$ and $\psi^*_{\alpha,b}$ with ψ_{FT} that is based on the FT-algorithm introduced in Abate and Valkó [18]. In this example (see Table 3) by taking the parameter $a = 4000$ in $\psi_{a,b}$ we reproduced four digits of the values derived by the FT-algorithm. By increasing a one can construct more accurate approximation of ψ using the SmartXML programming language and/or Mathematica package. In Table 4, several values of $\psi^*_{\alpha,b}$ and ψ_{FT} are compared when $a = 1000$.

Note that the algorithms written for evaluating the MR-approximations $\psi_{a,b}$ and $\psi^*_{\alpha,b}$ are not as fast as the one based on the FT algorithm, see the last columns of Tables 3 and 4 where the total execution times for each row are recorded.

The plot in Fig. 2(b) displays the difference $\Delta\psi(x_j) = \psi_{\alpha,b}(x_j) - \psi_{FT}(x_j)$ evaluated at $x_j = \log(\alpha/(\alpha - j + 1))/\log b$ for several values of $j = 500k$, $0 \le k \le 34$, when $\alpha = 4000$ and $b = 1.14795$.

Example 3

Finally, consider the case when the claim size distribution f is specified as log-normal (μ, σ) with $\mu = -1.62$ and $\sigma = 1.8$. Let us compare $\psi_{\alpha,b}$ with approximation ψ_{TWS} proposed in Trefethen et al. [9]. To approximate $\mathscr{L}_f$ we used the algorithm cordially provided by Hansjörg Albrecher that was applied in Albrecher et al. (2010) as well. Substitution of approximated values $\tilde{\mathscr{L}}_f$ of $\mathscr{L}_f$ into (16) and (17) yields the approximation $\tilde{\psi}_{\alpha,b}$. In Table 5, see below, we took $\alpha = 200$ and $b = 1.004158$. Several values of $\tilde{\psi}_{\alpha,b}$ as well as those of ψ_{TWS} are recorded when $\lambda = 1$ and $c \in \{1.05, 1.1, 1.15, 1.2, 1.25, 1.3, 2.0\}$. Here the values of ψ_{TWS} are borrowed from Table 9 in [10].

5. Conclusions

From the simulation study we conclude that the accuracy of the approximation $\psi_{\alpha,b}$ is considerably improved compared to the ones when the smaller values of parameter α are used. We also saw that the choice of optimal b depends on the behavior of ψ on the tail. Namely, the smaller values of b are recommended when ψ has a long tail. Besides, if α is larger than 60, then $\psi_{\alpha,b}$ performs better in terms of *sup*-norm when compared with the ME approximation ψ_{EME} with $M = 8$ fractional moments. Based on the model considered in Example 2, we see that both $\psi_{\alpha,b}$ and ψ_{FT} are comparable if α is considerably large. And lastly, note that performance of $\psi_{\alpha,b}$ is also comparable with approximation the ψ_{TWS} based on Trefethen–Weideman–Schmelzer method. Only the execution time is slow.

Finally, let us state several advantages of the MR-approximations $\psi_{\alpha,b}$ and $\psi^*_{\alpha,b}$: (a) they can be easily implemented; (b) in general, $\psi_{\alpha,b}$ performs better when compared to $\psi^*_{\alpha,b}$; (c) the proposed constructions are based on the knowledge of the Laplace transforms evaluated for only a finite number of positive arguments; (d) construction $\hat{\psi}_{\alpha,b}$ based on the empirical Laplace transform $\hat{\mathscr{L}}_f$ provides sufficiently good estimate of ψ; (e) in our calculations we were able to use the number $\alpha \ge 60$ combined with optimal $1 < b \le e$. As a result we obtained a more accurate approximation of the ruin probability ψ when compared to the MR-approximation based on [12] (i.e., when $b = e$).

The disadvantage of $\psi_{\alpha,b}(u)$ and $\psi^*_{\alpha,b}(u)$ is that for calculation of their values at very large initial capital $u > 0$, one should consider a sufficiently large value of α combined with a sufficiently small value of $b > 1$, such that $u \le \ln \alpha/\ln b$; the values of both $\psi_{\alpha,b}(u)$ and $\psi^*_{\alpha,b}(u)$ with $u > \ln \alpha/\ln b$ become small but constant. Besides, the evaluation time for $\psi^*_{\alpha,b}$ is too large when compared to other approaches considered in this work. It can be explained by presence of double summation in definition of $\psi^*_{\alpha,b}$.

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Appendix

Below we recall two statements established in Mnatsakanov [5,12].

Assume that for some $b \in (1, e]$:

$$M_k = \sup_{x \in \mathbb{R}_+} |f(x) b^{kx}|, \quad k = 1, 2 \text{ with } M_2 < \infty, \quad \text{and} \quad M_3 = \sup_{x \in R_+} |f'(x) b^{2x}| < \infty. \tag{A.1}$$

Let $F_{a,b} = 1 - \bar{F}_{a,b}$ with $\bar{F}_{a,b}$ defined in (3).

Theorem 1A ([5])

If the functions f and f′ are bounded on $\mathbb{R}_+$ and conditions (A.1) *are satisfied, then* $F_{\alpha,b} \to_u F$, *and*

$$\|F_{\alpha,b}-F\| \leq \frac{1}{\alpha+1}\left\{\frac{M_1}{\ln^2 b}+\frac{M_2}{2\ln b}+\frac{M_3}{2\ln^2 b}\right\}+o\left(\frac{1}{\alpha}\right) \quad as\ \alpha \to \infty.$$

Consider the moment sequence of a bivariate distribution F: $[0, 1]^2 \to [0, 1]$:

$$\mu_{j,m}(F)=\int_{[0,1]^2} t^j s^m dF(t,s):=(\mathscr{K}F)(j.m),\ \ (j,m)\in\mathbb{N}_a:=\mathbb{N}_\alpha\times\mathbb{N}_{\alpha'},\ \ \mu_{0,0}(F)=1.$$

Define the following transformations $\mathscr{K}_a^{-1}$ and $\mathscr{B}_a^{-1}\,\nu$ of $\nu = \{\mu_{j,m}, (j,m) \in \mathbb{N}_a\}$ by

$$\left(\mathscr{K}_a^{-1}\,\nu\right)(x,y)=\sum_{k=0}^{[\alpha x]}\sum_{l=0}^{[\alpha' y]}\sum_{j=k}^{\alpha}\sum_{m-l}^{\alpha'}\binom{\alpha}{j}\binom{j}{k}\binom{\alpha'}{m}\binom{m}{l}(-1)^{j+m-k-l}\mu_{j,m}(F), \quad \text{(A.2)}$$

and

$$f_{a,v}(x,y):=\mathscr{B}_a^{-1}\,\nu(x,y)=\frac{\Gamma(\alpha+2)\,\Gamma(\alpha'+2)}{\Gamma([\alpha x]+1)\,\Gamma([\alpha' y]+1)}$$
$$\times\sum_{m=0}^{\alpha-[\alpha x]}\sum_{j=0}^{\alpha'-[\alpha' y]}\frac{(-1)^{m+j}\mu_{m+[\alpha x],j+[\alpha' y]}(F)}{m!j!(\alpha-[\alpha x]-m)!(\alpha'-[\alpha' y]-j)!},\quad 0\leq x,y\leq 1, \quad \text{(A.3)}$$

where $0 \leq x, y \leq 1$, $a = (\alpha,\alpha')$ with $\alpha,\alpha' \in \mathbb{N}$.

Let us recover the bivariate distribution $F:\mathbb{R}_+^2 \to [0,1]$ and its density function f from knowledge of their Laplace–Stieltjes transform

$$L_F(j,m)=\int_{\mathbb{R}_+^2}[\phi_1(x)]^j[\phi_2(y)]^m dF(x,y), \quad (j,m)\in\mathbb{N}_a\times\mathbb{N}_{\alpha'}.$$

Here $\varphi_k(x) = e^{-x}$, $k = 1, 2$. The MR-constructions (A.2) and (A.3) can be applied to approximate the Laplace transform inversions in this case too. The following statement is valid.

Theorem 2A (Corollary 4 in [12])

Let $\nu = \{\mu_{j,m}(F) = L_F(j,m), (j,m) \in \mathbb{N}_a\}$.

- **i.** *If* $F_{a,\nu}=\mathscr{K}_a^{-1}\,\nu$, *then* $F_{a,\nu} \to wF_\nu$, *where* $F_\nu(x,y)=\int_{-\ln x}^{\infty}\int_{-\ln y}^{\infty}df(u.v)$, $x, y \in [0,1]$.
- **ii.** *If* $\bar{F}_{a,\nu}(x, y) = F_{a,\nu}(\varphi_1(x),\varphi_2(y))$, $x, y \in \mathbb{R}_+$, *then* $\bar{F}_{a,\nu} \to_w \bar{F}_\nu$, *as* $a \to \infty$, *where* $\bar{F}_\nu$ *is the survival function of F*, $\overline{F}_\nu(x,y)=\int_x^{\infty}\int_y^{\infty}df(u,v)$.

iii. *If* $f_{a,\nu}=\mathscr{B}_a^{-1}\,\nu$ *is defined by* (A.3) *and* $f^*_{a,\nu}(x,y)=f_{a,v}(\phi_1(x),\phi_2(y)e^{-(x+y)}$*, then* $f^*_{a,\nu}\to u$ *f on any compact in* $\mathbb{R}^2_+$.

Remark A1

If $\mathbf{1}$ = (1, 1), $\nu = \{\mu_{j,m} = L_F\,(j,m),\,(j,m) \in \mathbb{N}_a\}$, and the operator $\mathscr{B}_a^{-1}$ is defined according to (A.3), then to recover pdf f by means of L_F one can take:

$$(\mathscr{L}_a^{-1}\,\nu)\,(\mathbf{x})=(\mathscr{B}_a^{-1}\,\nu)\,(\phi_1(x),\phi_2(y))\,e^{-\mathbf{x}\cdot\mathbf{1}}, \quad \mathbf{x}=(x,y)\in\mathbb{R}^2_+.$$

For the univariate case, taking

$$\mu_j(F):=L_F(j)=\int_{\mathbb{R}_+}e^{-\tau\,j}\,dF(\tau), \quad j\in\mathbb{N}_\alpha,$$

we obtain the Laplace inversions for recovering f and $\bar{F} = 1 - F$, respectively:

$$(\mathscr{L}_\alpha^{-1}\,\nu)\,(x)=\frac{e^{-x}\,\Gamma(\alpha+2)}{\Gamma([\alpha e^{-x}]+1)}\sum_{m=0}^{\alpha-[\alpha e^{-x}]}\frac{(-1)^m L_F(m+[\alpha e^{-x}])}{m!(\alpha-[\alpha e^{-x}]-m)!} \quad \text{(A.4)}$$

$$(\overline{L}_\alpha^{-1}\,\nu)\,(x)=\sum_{k=0}^{[\alpha e^{-x}]}\sum_{j=k}^{\alpha}\binom{\alpha}{j}\binom{j}{k}(-1)^{j-k}L_F(j), \quad x\in\mathbb{R}_+. \quad \text{(A.5)}$$

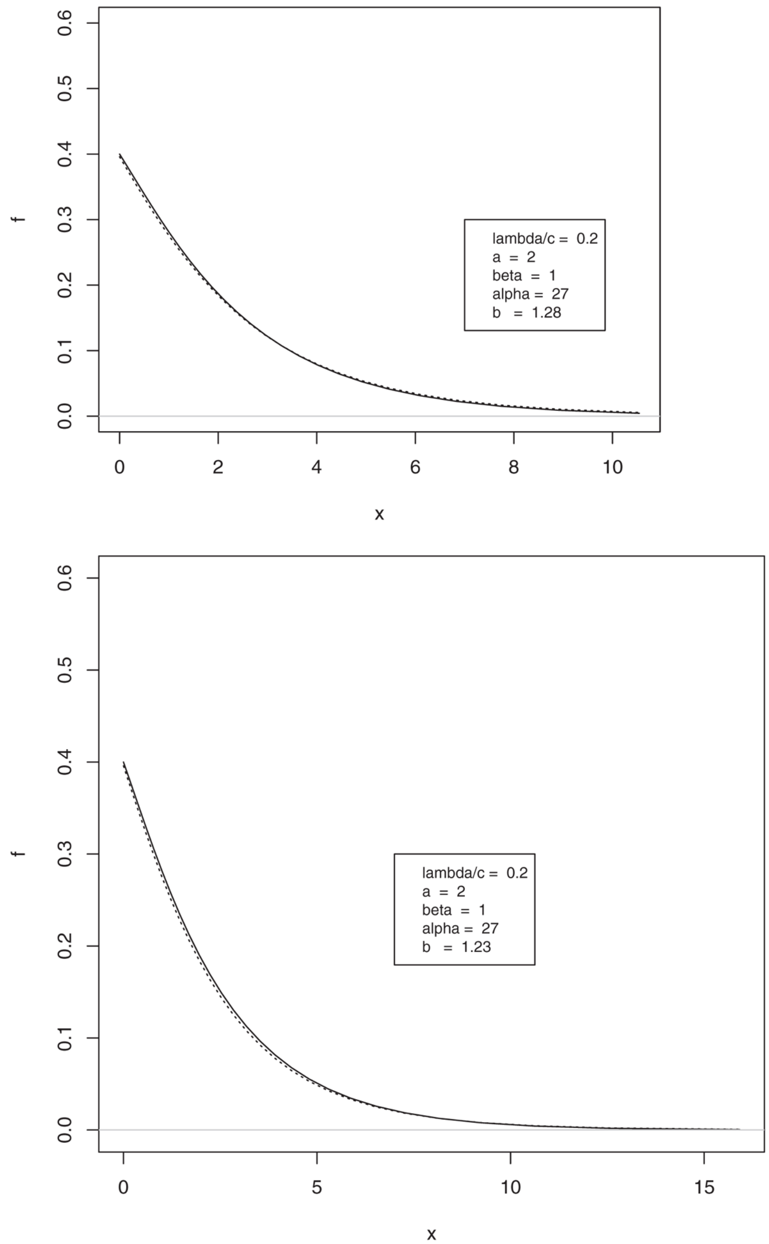
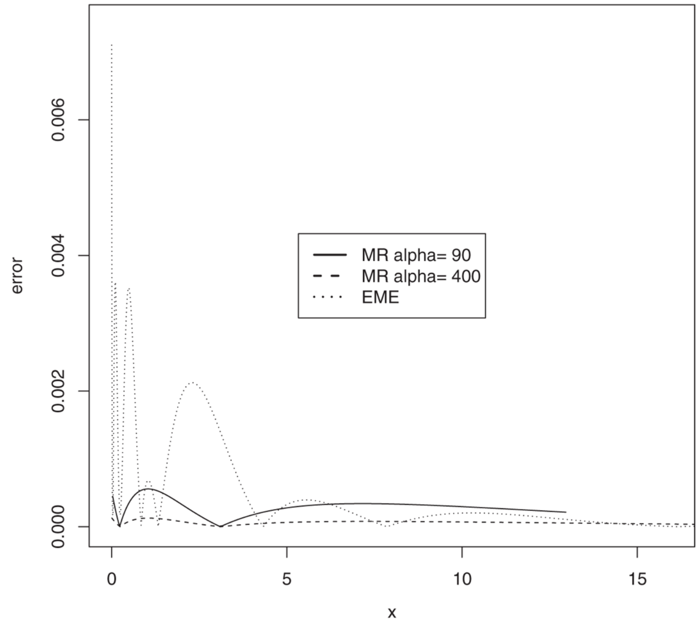


Fig. 1.

$X \sim$ gamma(2, 1), $\psi(x) = 0.461862\, e^{-0.441742\, x} - 0.061862\, e^{-1.358257\, x}$. (a) Approximated $\psi_{\alpha,b}$ (dotted) ruin probability when $b = 1.28$. (b) Estimated $\widehat{\psi_{\alpha,b}}$ (dotted) ruin probability when $b = 1.23$ and $n = 500$. In both plots $\alpha = 27$ and $\lambda_c = 0.2$.



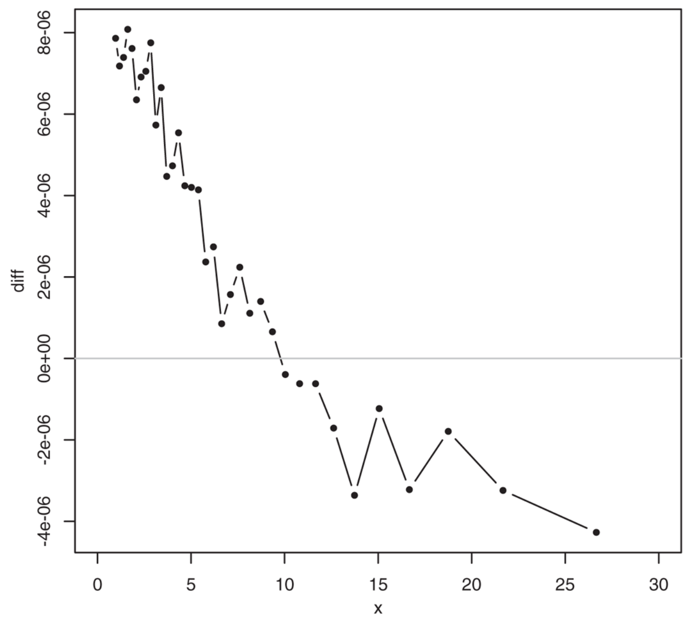


Fig. 2.

(a) X ~ gamma(2, 1). Absolute errors $|\psi_{\mathrm{EME}} - \psi|$ (dotted) and $|\psi_{\alpha,b} - \psi|$ (solid and slashed), when M = 8, α = 90, and α = 400, respectively. Here λ_c = 0.2, b = 1.415. (b) X ~ gamma(2.5, 0.4); difference $\psi_{\alpha,b} - \psi_{\mathrm{FT}}$ with α = 4000 and b = 1.14795. Here λ = 1 and c = 1.1.

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Table 1

Records of $10^4 \times \|\psi_{a,b} - \psi\|$ when $X \sim$ gamma(2, 1) and $\lambda_c = 0.2$.

$a \setminus b$	1.35	1.40	1.41	1.415	1.4175	1.42	1.425	1.43	1.45	1.50
60	15.40	9.64	8.72	8.29	8.08	8.25	8.83	9.40	11.55	16.18
90	10.33	6.47	5.86	5.57	5.59	5.79	6.17	6.55	7.97	11.04
120	7.78	4.88	4.42	4.20	4.31	4.46	4.74	5.02	6.09	8.37
150	6.23	3.91	3.54	3.39	3.50	3.62	3.85	4.07	4.92	6.74
200	4.68	2.94	2.66	2.58	2.67	2.76	2.93	3.10	3.73	5.09
400	2.35	1.47	1.34	1.32	1.36	1.41	1.50	1.58	1.90	2.57

Table 2

The values of ruin probabilities $\psi_{a,b}$ and ψ_{FT} evaluated at several values of $x_j = \log(a/(a - j + 1))/\log(b)$. Here $X \sim$ gamma (2, 1), $a = 5000$, $b = 1.4125$, $\lambda = 1$, and $c = 5$.

j	500	600	700	800	900	1000	Time (s)
$\psi_{a,b}(x_j)$	0.362832	0.354852	0.346722	0.33845	0.330054	0.321537	0.56173
$\psi_{FT}(x_j)$	0.362834	0.354856	0.346728	0.33846	0.330062	0.321546	0.03704
j	$2. \times 10^3$	2.5×10^2	$3. \times 10^3$	3.5×10^2	$4. \times 10^3$	4.5×10^2	
$\psi_{a,b}(x_j)$	0.232083	0.186348	0.141461	0.098558	0.058918	0.024351	4.71034
$\psi_{FT}(x_j)$	0.232092	0.186355	0.141463	0.098556	0.058913	0.024345	0.04071

Table 3

The values of ruin probabilities $\psi_{a,b}$ and ψ_{FT} evaluated at several values of $x_j = \log(a/(a - j + 1))/\log(b)$. Here $X \sim$ gamma (2.5, 0.4), $a = 4000$, $b = 1.14795$, $\lambda = 1$, and $c = 1.1$.

j	500	600	700	800	900	1000	Time (s)
$\psi_{a,b}(x_j)$	0.811541	0.789650	0.767593	0.745439	0.723222	0.700960	0.358847
$\psi_{FT}(x_j)$	0.811533	0.789643	0.767586	0.745431	0.723215	0.700954	0.052368
j	1.5×10^2	2×10^3	2.5×10^2	3×10^3	3.5×10^2	4×10^3	
$\psi_{a,b}(x_j)$	0.589077	0.476154	0.361914	0.245881	0.127029	0.000330	2.411974
$\psi_{FT}(x_j)$	0.589071	0.476150	0.361913	0.245882	0.127031	0.000337	0.067436

Table 4

The values of ruin probabilities $\psi^*_{\alpha,b}$ and ψ_{FT} evaluated at several values of $x_j = \log(\alpha/(\alpha - j + 1))/\log(b)$. Here $X \sim$ gamma (2.5, 0.4), $\alpha = 1000$, $b = 1.1485$, $\lambda = 1$, and $c = 1.1$.

j	25	50	100	200	300	400	Time (s)
$\psi^*_{\alpha,b}$	0.892210	0.873740	0.832823	0.745543	0.656748	0.567289	96.774
$\psi_{FT}(x_j)$	0.893504	0.875025	0.834044	0.746643	0.657754	0.568208	0.051
j	500	600	700	800	900	1000	
$\psi^*_{\alpha,b}$	0.477094	0.386005	0.293782	0.200016	0.103831	0.00120577	41.896
$\psi_{FT}(x_j)$	0.477920	0.386737	0.294416	0.200546	0.104238	0.00129338	0.0534

Table 5

The values of ruin probabilities $\psi_{\mathrm{TWS}}(x)$ and $\tilde{\psi}_{\alpha,b}(x)$ evaluated at $x = 100$ and $x = 1000$. Here $X \sim$ log-normal $(-1.62, 1.8)$, $\alpha = 200$, $b = 1.004158$, $\lambda = 1$, and $c \in \{1.05, 1.1, 1.15, 1.20, 1.25, 1.30, 2.0\}$.

c	**1.05**	**1.1**	**1.15**	**1.2**	**1.25**	**1.3**	**2.0**
x				$x = 100$			
$\tilde{\psi}_{\alpha,b}$	0.550841	0.343504	0.235396	0.172918	0.133785	0.107664	0.0254206
ψ_{TWS}	0.550743	0.343954	0.235726	0.173086	0.133839	0.107647	0.0253454
x				$x = 1000$			
$\tilde{\psi}_{\alpha,b}$	0.0397047	0.0104754	0.0054816	0.0036679	0.0027503	0.0021988	0.0005765
ψ_{TWS}	0.0419949	0.0109919	0.0057413	0.0038406	0.0028796	0.0023021	0.0006037